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# Nova ratio quantitates irrationales proxime exprimendi

Leonhard Euler

# NOVA RATIO QVANTITATES IRRATIONALES PROXIME EXPRIMENDI.

Auctore
*L. EVLERO.*

1.

Omnem quantitatem irrationalem simplicem ad hanc formam $(1+x)^n$ reduci posse constat siquidem exponens $n$ numerum quemcunque fractum designare assumatur; quicunque enim numerus N ad exponentem fractum $n = \frac{\mu}{\nu}$ eleuandus proponatur eum semper ad hanc formam $a^\nu + b$ reducare licet vnde formula proposita fit $(a^\nu + b)^{\frac{\mu}{\nu}} = a^\mu (1 + \frac{b}{a^\nu})^{\frac{\mu}{\nu}}$ sicque irrationalibus continetur in expressione $(1 + \frac{b}{a^\nu})^{\frac{\mu}{\nu}}$ quae cum formula proposita $(1+x)^n$ congruit ponendo $\frac{b}{a^\nu} = x$ et $\frac{\mu}{\nu} = n$. Ac si pro $a$ fractiones velimus admittere, ac $b$ aeque negatiue ac positiue sumere, quantitas $\frac{b}{a^\nu}$ hoc modo iam quouis casu satis parua effici potest, vnde etiam more consueto formula $(1+x)^n$ in seriem admodum conuergentem resoluitur.

2. Per

2. Per euolutionem ſcilicet binomii Neutonianam haec formula $(1+x)^n$ duplici modo in ſeriem infinitam conuertitur, primum nempe directe:

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}$$

tum vero quia eſt $(1+x)^n = \frac{1}{(1+x)^{-n}}$ erit quoque

$$(1+x)^n = \frac{1}{1 - \frac{n}{1}x + \frac{n(n+1)}{1 \cdot 2}x^2 - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}}$$

Hinc vero porro has expreſſiones inuicem multiplicando, et pro $2n$ ſcribendo $n$ deriuabitur tertia expreſſio multo magis conuergens:

$$(1+x)^n = \frac{1 + \frac{n}{2}x + \frac{n(n-2)}{2 \cdot 4}x^2 + \frac{n(n-2)(n-4)}{2 \cdot 4 \cdot 6}x^3 + \text{etc.}}{1 - \frac{n}{2}x + \frac{n(n+2)}{2 \cdot 4}x^2 - \frac{n(n+2)(n+4)}{2 \cdot 4 \cdot 6}x^3 + \text{etc.}}$$

3. Attendenti autem facile patebit, infinitas expreſſiones huic poſtremae ſimiles exhiberi poſſe, quae ſingulae aequales ſint formulae propoſitae $(1+x)^n$, ſi enim ponamus:

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + etc.}{1 - \alpha x + \beta x^2 - \gamma x^3 + \delta x^4 - \varepsilon x^5 + \zeta x^6 - etc.}$$

determinatio coëfficientium praebet problema indeterminatum, atque adeo ſi vel numerator vel denominator ad lubitum aſſumitur, alterius coëfficientes inde determinantur. Hinc quaeſtio naſcitur maximi momenti, quomodo tam numerator quam denominator determinari debeant, ut ambo ſimul maxime conuergant: atque hic quidem denominatori finitum terminorum numerum tribuere licet, vbi quaeſtio huc redit, quomodo coëfficientes denominatoris assumi oporteat, vt pro numeratore resulte series maxime conuergens.

4. Quodsi autem in denominatore datus terminorum numerus constituatur, numerator erit series maxime conuergens, si vnus pluresue eius termini se ordine excipientes plane euanescant; tum enim sequentes termini tam fient exigui, si quidem fuerit $x < 1$, vt sine notabili errore reiic queant. Atque hic notari conuenit, si pro denominatore sumatur binomium $1 - \alpha x$, quemlibet numeratoris terminum ad nihilum redigi posse; sin autem denominator statuatur trinomium, bini termini successiui numeratoris in nihilum redigi poterunt; terni vero et ita porro, si pro denominatore quadrinomium vel multinomium assumatur. Tum vero etiam perspicuum est aduergentiam eo fore maiorem, quo longius numeratoris termini euanescentes ab initio distent; vnde sequentia problemata resoluenda occurrunt.

## Problema I.

5. *Binomii potestatem $(1 + x)^n$ transformare in talem expressionem maxime conuergentem:*

$$(1 + x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + etc.}{1 - \alpha x}$$

*denominatore existente binomio.*

## Solutio.

Si potestas $(1 + x)^n$ in seriem euoluatur, eaque per denominatorem $1 - \alpha x$ multiplicetur, orietur sequens aequatio conficienda:

6 = 1

$$0 = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.\,2}x^2 + \frac{n(n-1)(n-2)}{1.\,2.\,3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}x^4 + \text{etc.}$$
$$-\alpha - \frac{n}{1}\alpha - \frac{n(n-1)}{1.\,2}\alpha - \frac{n(n-1)(n-2)}{1.\,2.\,3}\alpha - \text{etc.}$$
$$-1 - A - B - C - D - \text{etc.}$$

Iam prouti numeratoris terminus vel secundus vel tertius vel quartus etc. euanescere debet, sequentes coëfficientium determinationes obtinebuntur:

I. Si $A = 0$, habetur statim $\alpha = \frac{n}{1}$; et sequentes numeratoris termini erunt:

$$B = -\frac{n(n+1)}{1.\,2};\quad C = -\frac{2n(n-1)(n+1)}{1.\,2.\,3};\quad D = -\frac{3n(n-1)(n-2)(n+1)}{1.\,2.\,3.\,4}\ \text{etc.}$$

II. Si $B = 0$, habetur statim $\alpha = \frac{n-1}{2}$, et pro numeratore:

$$A = \frac{n+1}{1.\,2};\quad C = -\frac{1(n+1)n(n-1)}{2.\,1.\,2.\,3};\quad D = -\frac{2(n+1)n(n-1)(n-2)}{2.\,1.\,2.\,3.\,4}\ \text{etc.}$$

III. Si $C = 0$; habetur $\alpha = \frac{n-2}{3}$ et pro numeratore:

$$A = \frac{2(n+1)}{3.\,1};\quad B = \frac{1(n+1)n}{3.\,1.\,2};\quad D = -\frac{1(n+1)n(n-1)(n-2)}{3.\,1.\,2.\,3.\,4}\ \text{etc.}$$

IV. Si $D = 0$; habetur $\alpha = \frac{n-3}{4}$ et pro numeratore:

$$A = \frac{3(n+1)}{4.\,1};\quad B = \frac{2(n+1)n}{4.\,1.\,2};\quad C = \frac{1(n+1)n(n-1)}{4.\,1.\,2.\,3}\ \text{etc.}$$

Hinc iam in genere patet, si quilibet alius sequentium terminorum in numeratore debeat euanescere, haberi primo:

$$\alpha = \frac{n-\omega}{\omega+1}\ \text{et pro numeratore:}$$

$$A = \frac{\omega(n+1)}{(\omega+1)1};\quad B = \frac{(\omega-1)}{(\omega+1)}\frac{(n+1)n}{1.\,2};\quad C = \frac{(\omega-2)}{(\omega+1)}\frac{(n+1)n(n-1)}{1.\,2.\,3}$$
$$D = \frac{(\omega-3)}{(\omega+1)}\frac{(n+1)n(n-1)(n-2)}{1.\,2.\,3.\,4};\quad E = \frac{(\omega-4)}{\omega+1}.\ \frac{(n+1)n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4.\,5}$$

 cuius

cuius progressionis lex est manifesta.

## Coroll. 1.

6. Quodsi iam in numeratore termini, qui euanescentem sequuntur, omittantur, habebuntur expressiones finitae ac rationales continuo propius valorem $(1+x)^n$ exhibentes; ita si primo ponatur $A=0$, habebitur ista approximatio:

$$(1+x)^n = \frac{1}{1-nx}$$

quae etsi a veritate parum recedit, tamen magis aberrat quam sequentes.

## Coroll. 2.

7. Sit $B=0$, et secundus casus praebebit hanc approximationem:

$$(1+x)^n = \frac{1+\frac{n+1}{2}x}{1-\frac{(n-1)}{2}x} = \frac{1+\frac{(n+1)}{2}x}{1-\frac{(n-1)}{2}x}.$$

Hinc si sit $n=\frac{\mu}{\nu}$ erit:

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1+\frac{(\mu+\nu)}{2\nu}x}{1-\frac{(\mu-\nu)}{2\nu}x}.$$

## Coroll. 3.

8. Sit $C=0$, et tertius casus dabit:

$$(1+x)^n = \frac{1+\frac{2(n+1)}{n}x+\frac{(n+1)n}{2}x^2}{\text{[illegible]}}$$

vnd

vnde si fuerit $n = \frac{\mu}{\nu}$ erit:

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1 + \frac{2(\mu+\nu)}{3 \cdot 1 \cdot \nu} x + \frac{1(\mu+\nu)\mu}{3 \cdot 1 \cdot 2 \nu^2} x^2}{1 - \frac{(\mu - 2\nu)}{3\nu} x}.$$

## Coroll. 4.

9. Sit $D = 0$, et quartus casus dat:

$$(1+x)^n = \frac{1 + \frac{3(n+1)}{4 \cdot 1} x + \frac{2(n+1)n}{4 \cdot 1 \cdot 2} x^2 + \frac{1(n+1)n(n-1)}{4 \cdot 1 \cdot 2 \cdot 3} x^3}{1 - \frac{(n-3)}{4} x}$$

ideoque si $n = \frac{\mu}{\nu}$ erit:

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1 + \frac{3(\mu+\nu)}{4 \cdot 1 \nu} x + \frac{2(\mu+\nu)\mu}{4 \cdot 1 \cdot 2 \nu^2} x^2 + \frac{1(\mu+\nu)\mu(\mu-\nu)}{4 \cdot 1 \cdot 2 \cdot 3 \nu^3} x^3}{1 - \frac{(\mu - 3\nu)}{4\nu} x}$$

vnde perspicuum est, quomodo huiusmodi formulae vlterius continuari debent; quamobrem plures hic non exhibeo.

## Coroll. 5.

10. In genere autem habebitur haec forma:

$$(1+x)^n = \frac{1 + \frac{(\omega-1)(n+1)}{\omega \cdot 1} x + \frac{(\omega-2)(n+1)n}{\omega \cdot 1 \cdot 2} x^2 + \frac{(\omega-3)(n+1)n(n-1)}{\omega \cdot 1 \cdot 2 \cdot 3} x^3 + \text{etc.}}{1 - \frac{(n - \omega + 1)}{\omega} x}$$

vbi pro $\omega$ sumi potest numerus quicunque; haecque expressio si in infinitum continuetur, non solum ad veritatem appropinquat, sed ipsum verum valorem formulae $(1+x)^n$ exhibebit.

## Coroll. 6.

11. Si sumatur $\omega = n + 1$ denominator in vnitatem abibit, orieturque nota series Neutoniana:

 $(1+x)$

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.\,2}x^2 + \frac{n(n-1)(n-2)}{1.\,2.\,3}x^3 + \text{etc.}$$

Sin autem pro $\omega$ capiatur numerus infinitus, erit:

$$(1+x)^n = \frac{1 + \frac{(n+1)}{1}x + \frac{(n+1)n}{1.\,2}x^2 + \frac{(n+1)n(n-1)}{1.\,2.\,3}x^3 + \text{etc.}}{1+x}$$

cuius ratio quoque ex binomio Neutoniano est manifesta.

## Coroll. 7.

12. Si ponatur $\omega = n$ habebitur: $(1+x)^n =$

$$\frac{1 + \frac{(n+1)(n-1)}{1.\,n}x + \frac{(n+1)(n-2)}{1.\,2}x^2 + \frac{(n+1)(n-1)(n-3)}{1.\,2.\,3}x^3 + \frac{(n+1)(n-1)(n-2)(n-4)}{1.\,2.\,3.\,4}x^4 + \text{etc.}}{1 - \frac{1}{n}x}$$

vel numeratorem et denominatorem per $n$ multiplicando:

$$(1+x)^n = \frac{n + \frac{(n+1)(n-1)}{1}x + \frac{(n+1)n(n-2)}{1.\,2}x^2 + \frac{(n+1)n(n-1)(n-3)}{1.\,2.\,3}x^3 + \text{etc.}}{n - x}$$

## Coroll. 8.

13. Si ponatur $\omega = x$, fiet denominator $= x - n$ et obtinetur: $(1+x)^n =$

$$\frac{1 + \frac{(n+1)}{1}(x-1) + \frac{(n+1)n}{1.\,2}x(x-2) + \frac{(n+1)n(n-1)}{1.\,2.\,3}x^2(x-3) + \frac{(n+1)n(n-1)(n-2)}{1.\,2.\,3.\,4}x^3(x-4) + \text{etc.}}{x - n}$$

similique modo ex hac expressione innumerabiles series deduci possunt, quarum ratio aliunde non tam facile perspici poterit; vnde haec inuestigatio doctrinam serierum non mediocriter amplificare videtur.

Proble-

## Problema II.

14. *Binomii poteſtatem* $(1+x)^n$ *transformare in huiusmodi ſeriem maxime conuergentem:*

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}}{1 - \alpha x + \beta x^2}$$

*denominatore exiſtente trinomio.*

## Solutio.

Reſoluta poteſtate $(1+x)^n$ in ſeriem more conſueto, confici oportebit ſequentem aequationem:

$$\begin{aligned} 0 = 1 &+ \tfrac{n}{1}x + \tfrac{n(n-1)}{1.\,2}x^2 + \tfrac{n(n-1)(n-2)}{1.\,2.\,3}x^3 + \tfrac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}x^4 + \text{etc.}\\ &- \alpha - \tfrac{n}{1}\alpha - \tfrac{n(n-1)}{1.\,2}\alpha - \tfrac{n(n-1)(n-2)}{1.\,2.\,3}\alpha - \text{etc.}\\ &\qquad + \beta + \tfrac{n}{1}\beta + \tfrac{n(n-1)}{1.\,2}\beta + \text{etc.}\\ -1 &- A - B - C - D - \text{etc.} \end{aligned}$$

atque hic denominatorem $1 - \alpha x + \beta x^2$ ita definire licet, vt in numeratore bini termini ſucceſſiue euaneſcant, vnde is eo magis conuergens reddetur:

I. Sit $A = 0$ et $B = 0$, erit $\alpha = \frac{n}{1}$ et $\beta = \frac{n(n+1)}{1.\,2}$, vnde habetur

$$C = \tfrac{n}{1}\left(\tfrac{(n-1)(n-2)}{2.\,3} - \tfrac{(n-1)n}{2.\,1} + \tfrac{(n+1)n}{1.\,2}\right) = \tfrac{1(n+2)(n+1)n}{3.\,1.\,2.\,1}$$

$$D = \tfrac{n(n-1)}{1.\,2}\left(\tfrac{(n-1)(n-3)}{3.\,4} - \tfrac{(n-2)n}{3.\,1} + \tfrac{n(n+1)}{1.\,2}\right) = \tfrac{2(n+2)(n+1)n(n-1)}{4.\,1.\,2.\,3}$$

$$E = \tfrac{n(n-1)(n-2)}{1.\,2.\,3}\left(\tfrac{(n-3)(n-4)}{4.\,5} - \tfrac{(n-3)n}{4.\,1} + \tfrac{(n+1)n}{1.\,2}\right) = \tfrac{3(n+2)(n+1)n(n-1)(n-2)}{5.\,1.\,2.\,3.\,4}$$

et in genere erit:

$$N = \ldots\ldots\left(\tfrac{(n-\nu)(n-\nu-1)}{(\nu+1)(\nu+2)} - \tfrac{(n-\nu)n}{(\nu+1)1} + \tfrac{(n+1)n}{1.\,2}\right) = \ldots\ldots\tfrac{\nu(n+2)(n+1)}{1.\,2.\,(\nu+2)}$$

ex quo generali valore illi ſpeciales facile deriuantur.

II.

II. Sit $B = 0$ et $C = 0$, erit pro $\alpha$ et $\mathfrak{B}$:

$$\mathfrak{B} - \frac{n}{1}\alpha + \frac{n(n-1)}{1\cdot 2} = 0; \qquad \alpha = \frac{2(n-1)}{3}$$
$$\mathfrak{B} - \frac{(n-1)}{2}\alpha + \frac{(n-1)(n-2)}{2\cdot 3} = 0; \quad \text{hinc} \quad \mathfrak{B} = \frac{n(n+1)}{2\cdot 3}.$$

Pro numeratore vero habebitur:

$$A = \frac{n}{1} - \frac{2(n-1)}{3} = \frac{n+2}{3}$$
$$D = \frac{n(n-1)}{1\cdot 2}\left(\frac{(n-2)(n-3)}{3\cdot 4} - \frac{2(n-2)(n-1)}{3\cdot 3} + \frac{n(n-1)}{[illegible]}\right) = [illegible]\frac{(n+2)(n+1)n(n-1)}{[illegible]\;3\cdot 1\cdot 2}$$
$$E = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{(n-3)(n-4)}{[illegible]} - \frac{2(n-3)(n-2)}{[illegible]} + \frac{n(n-1)}{3\cdot 3}\right) = \frac{2\cdot 3(n+2)(n+1)n(n-1)(n-2)}{4\cdot 5\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}$$
$$F = \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{(n-4)(n-5)}{[illegible]\;6} - \frac{2(n-4)(n-1)}{3\cdot 3} + \frac{n(n-1)}{2\cdot 3}\right) = \frac{3\cdot 4(n+2)(n+1)n(n-1)(n-2)(n-3)}{5\cdot 6\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3\cdot 4}$$

et in genere:

$$N = \ldots\ldots\left(\frac{(n-\nu)(n-\nu-1)}{(\nu+1)(\nu+2)} - \frac{2(n-\nu)(n-1)}{(\nu+1)\;3} + \frac{n(n-1)}{2\cdot 3}\right)\ldots\ldots = \frac{\nu(\nu-1)}{(\nu+1)(\nu+2)}\cdot\frac{(n+2)(n+1)}{2\cdot 3}[illegible]$$

III. Sit $C = 0$ et $D = 0$, at pro denominatore erit [illegible]

$$\mathfrak{B} - \frac{(n-2)}{2}\alpha + \frac{(n-2)(n-3)}{[illegible]} = 0; \qquad \alpha = \frac{n-3}{[illegible]}$$
$$\mathfrak{B} - \frac{(n-3)}{3}\alpha + \frac{(n-3)(n-4)}{3\cdot 4} = 0 \quad \text{hinc} \quad \mathfrak{B} = \frac{(n-1)(n-2)}{3\cdot 4}$$

hincque pro numeratore:

$$A = \frac{n}{1} - \frac{(n-2)}{2} = \frac{n+2}{2}$$
$$B = \frac{n(n-1)}{1\cdot 2} - \frac{n(n-2)}{[illegible]} + \frac{(n-1)(n-2)}{[illegible]} = \frac{(n+2)(n+1)}{[illegible]}$$
$$E = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{(n-3)(n-4)}{[illegible]} - \frac{(n-3)(n-2)}{[illegible]\;4} + \frac{(n-1)(n-2)}{3\cdot 4}\right) = \frac{(n+2)(n+1)n(n-1)(n-2)}{2\cdot 3\cdot 4\cdot 5\cdot 1\cdot 2\cdot 3}$$

Quia autem sufficit terminos, qui euanescentes antecedant, nosse, sequentes non determino, quia eorum lex deinceps patebit.

IV. Sit $D = 0$ et $E = 0$, erit pro denominatore

$$\mathfrak{B} - \frac{(n-2)}{[illegible]}\alpha + \frac{(n-2)(n-3)}{[illegible]} = 0; \qquad \alpha = \frac{2(n-3)}{4}$$
$$\mathfrak{B} - \frac{(n-3)}{4}\alpha + \frac{(n-3)(n-4)}{4\cdot 5} = 0 \quad \text{hinc} \quad \mathfrak{B} = \frac{(n-2)(n-3)}{4\cdot 5}$$

at pro numeratore reperietur:

$$A = \frac{n}{1} - \frac{2(n-3)}{5} = \frac{3(n+2)}{5}$$

$$B = \frac{n(n-1)}{1.\,2} - \frac{2n(n-3)}{1.\,5} + \frac{(n-2)(n-3)}{4.\,5} = \frac{3(n+2)(n+1)}{4.\,5}$$

$$C = \frac{n}{1}\left(\frac{(n-1)(n-2)}{2.\,3} - \frac{2(n-1)(n-3)}{2.\,5} + \frac{(n-2)(n-3)}{4.\,5}\right) = \frac{(n+2)(n+1)n}{5.\,4.\,3}.$$

V. Sit $E = 0$ et $F = 0$, atque ex allatis facile concludimus fore primo

$$\alpha = \frac{2(n-4)}{6};\; \beta = \frac{(n-3)(n-4)}{5.\,6} \text{ tum vero}$$

$$A = \frac{4(n+2)}{6};\; B = \frac{6(n+2)(n+1)}{6.\,5};\; C = \frac{4(n+2)(n+1)(n+0)}{6.\,5.\,4} \text{ et}$$

$$D = \frac{1(n+2)(n+1)(n+0)(n-1)}{6.\,5.\,4.\,3}.$$

Generaliter ergo denique has eliciemus determinationes:

$$\alpha = \frac{2(n-\omega)}{\omega+2};\quad \beta = \frac{(n-\omega)(n-\omega+1)}{(\omega+2)(\omega+1)}$$

$$A = \frac{\omega}{1}.\frac{(n+2)}{\omega+2}$$

$$B = \frac{\omega(\omega-1)}{1.\,2}.\frac{(n+2)(n+1)}{(\omega+2)(\omega+1)}$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3}.\frac{(n+2)(n+1)n}{(\omega+2)(\omega+1)\omega}$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{1.\,2.\,3.\,4}.\frac{(n+2)(n+1)n(n-1)}{(\omega+2)(\omega+1)\omega(\omega-1)}$$

$$E = \frac{\omega(\omega-1)(\omega-2)(\omega-3)(\omega-4)}{1.\,2.\,3.\,4.\,5}.\frac{(n+2)(n+1)n(n-1)(n-2)}{(\omega+2)(\omega+1)\omega(\omega-1)(\omega-2)}$$

etc.

vnde etiam coefficientes terminorum post euanescentes sequentium facile formantur.

## Coroll. I.

15. Quando pro denominatore in genere est:

$$\alpha = \frac{2(n-\omega)}{\omega+2} \quad\text{et}\quad \beta = \frac{(n-\omega)(n-\omega+1)}{(\omega+2)(\omega+1)}$$

pro numeratore habebimus:

$$A = \frac{\omega}{\omega + 2} \cdot \frac{n+2}{1}$$

$$B = \frac{\omega(\omega - 1)}{(\omega + 2)(\omega + 1)} \cdot \frac{(n+2)(n+1)}{1 \cdot 2}$$

$$C = \frac{(\omega - 1)(\omega - 2)}{(\omega + 2)(\omega + 1)} \cdot \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3}$$

$$D = \frac{(\omega - 2)(\omega - 3)}{(\omega + 2)(\omega + 1)} \cdot \frac{(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$E = \frac{(\omega - 3)(\omega - 4)}{(\omega + 2)(\omega + 1)} \cdot \frac{(n+2)(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

etc.

quorum valorum analogia ad eos, qui in primo problemate sunt inuenti, iam satis luculenter ordinem sequentium, vbi denominator pluribus constabit terminis, declarat.

## Coroll. 2.

16. Neglectis terminis in numeratore post euanescentes sequentibus, habebimus approximationes sequentes:

$$\text{si } \omega = 0 \text{ erit } (1+x)^n = \frac{1}{1 - nx + \frac{n(n+1)}{1 \cdot 2} xx}$$

quae quidem in hoc genere plurimum a veritate discrepat.

## Coroll. 3.

17. Ponamus $\omega = 1$, eritque proxime:

$$(1+x)^n = \frac{1 + \frac{n+1}{2} x}{1 - \frac{(n-1)}{2} x + \frac{n(n-1)}{2 \cdot 3} xx}$$

sin autem $\omega = 2$ erit adhuc propius:

$$(1+x)^n = \frac{1 + \frac{2(n+2)}{4}x + \frac{(n+2)(n+1)}{4.\ 3}xx}{1 - \frac{2(n-2)}{4}x + \frac{(n-2)(n-1)}{4.\ 3}xx}$$

et si $\omega = 3$ erit

$$(1+x)^n = \frac{1 + \frac{3(n+2)}{5}x + \frac{3(n+2)(n+1)}{5.\ 4}xx + \frac{(n+2)(n+1)n}{5.\ 4.\ 3}x^3}{1 - \frac{2(n-3)}{5}x + \frac{(n-3)(n-2)}{5.\ 4}xx}.$$

si $\omega = 4$ erit

$$(1+x)^n = \frac{1 + \frac{4(n+2)}{6}x + \frac{6(n+2)(n+1)}{6.\ 5}xx + \frac{4(n+2)(n+1)n}{6.\ 5.\ 4}x^3 + \frac{(n+2)(n+1)n(n-1)}{6.\ 5.\ 4.\ 3}x^4}{1 - \frac{2(n-4)}{6}x + \frac{(n-4)(n-3)}{6.\ 5}xx}$$

si $\omega = 5$ erit $(1+x)^n =$

$$\frac{1 + \frac{5(n+2)}{7}x + \frac{10(n+2)(n+1)}{7.\ 6}x^2 + \frac{10(n+2)(n+1)n}{7.\ 6.\ 5}x^3 + \frac{5(n+2)(n+1)n(n-1)}{7.\ 6.\ 5.\ 4}x^4 + \frac{(n+2)(n+1)n(n-1)(n-2)}{7.\ 6.\ 5.\ 4.\ 3}x^5}{1 - \frac{2(n-5)}{7}x + \frac{(n-5)(n-4)}{7.\ 6}xx}$$

Quae expressiones ex coefficientibus potestatum binomii expedite vlterius continuantur. Quo longius vero continuantur, eo minus a veritate aberrabunt.

## Coroll. 4.

18. Generaliter autem hanc formulae $(1+x)^n$ transformationem commodius exhibere non licet, quam vt dicamus esse

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + etc.}{1 - \alpha x + \beta xx}$$

existentibus coefficientium valoribus:

$$A = \frac{(\omega+1)\omega}{(\omega+2)(\omega+1)} \cdot \frac{n+2}{1} \qquad \alpha = \frac{2(n-\omega)}{\omega+2}$$
$$B = \frac{\omega(\omega-1)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)}{1 \cdot 2} \qquad \beta = \frac{(n-\omega)(n-\omega-1)}{(\omega+2)(\omega+1)}$$
$$C = \frac{(\omega-1)(\omega-2)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3}$$
$$D = \frac{(\omega-2)(\omega-3)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4}$$
$$E = \frac{(\omega-3)(\omega-4)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$
etc.

## Coroll. 5.

19. Hic iterum patet, cum quantitas $\omega$ ab arbitrio nostro pendeat, si capiatur $\omega = n$, prodire $\alpha = 0$, $\beta = 0$ et

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}$$

Sin autem sit $\omega = \infty$ erit $\alpha = -2$, et $\beta = 1$; vnde

$$(1+x)^n = \frac{1 + \frac{n+2}{1}x + \frac{(n+2)(n+1)}{1 \cdot 2}x^2 + \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}}{1 + 2x + xx}$$

seu $(1+x)^n = \frac{(1+x)^{n+2}}{(1+x)^2}$, cuius ratio est manifesta.

## Problema III.

20. *Binomii potestatem* $(1+x)^n$ *transformare in huiusmodi seriem maxime conuergentem:*

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}}{1 - \alpha x + \beta x^2 - \gamma x^3}$$

*denominatore existente quadrinomio.*

Solutio.

## Solutio.

Sequens ergo aequatio construi debet:

$$\begin{array}{llllll}
0 = 1 & + \frac{n}{1}x & + \frac{n(n-1)}{1\cdot 2}x^2 & + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 & + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 & + \text{etc.} \\
 & - \alpha & - \frac{n}{1}\alpha & - \frac{n(n-1)}{1\cdot 2}\alpha & - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\alpha & - \text{etc.} \\
 & & + \beta & + \frac{n}{1}\beta & + \frac{n(n-1)}{1\cdot 2}\beta & + \text{etc.} \\
 & & & - \gamma & - \frac{n}{1}\gamma & - \text{etc.} \\
-1 & - A & - B & - C & - D & - \text{etc.}
\end{array}$$

Hic iam effici potest, vt in serie coefficientium A, B, C, D etc. terni successiui euanescant: Sumantur ergo terni quicunque successiue euanescentes, et obtinebuntur tres huiusmodi aequationes:

$$\gamma - \frac{(n-\omega+2)}{\omega-1}\beta + \frac{(n-\omega+2)(n-\omega+1)}{(\omega-1)(\omega+0)}\alpha - \frac{(n-\omega+2)(n-\omega+1)(n-\omega)}{(\omega-1)(\omega-0)(\omega+1)} = 0$$

$$\gamma - \frac{(n-\omega+1)}{\omega}\beta + \frac{(n-\omega+1)(n-\omega)}{\omega(\omega+1)}\alpha - \frac{(n-\omega+1)(n-\omega)(n-\omega-1)}{\omega(\omega+1)(\omega+2)} = 0$$

$$\gamma - \frac{(n-\omega)}{\omega+1}\beta + \frac{(n-\omega)(n-\omega-1)}{(\omega+1)(\omega+2)}\alpha - \frac{(n-\omega)(n-\omega-1)(n-\omega-2)}{(\omega+1)(\omega+2)(\omega+3)} = 0.$$

Hinc differentiis sumendis habebitur:

$$\frac{(n+1)}{(\omega-1)\omega}\beta - \frac{2(n+1)(n-\omega+1)}{(\omega-1)\omega(\omega+1)}\alpha + \frac{3(n+1)(n-\omega+1)(n-\omega)}{(\omega-1)\omega(\omega+1)(\omega+2)} = 0$$

$$\frac{(n+1)}{\omega(\omega+1)}\beta - \frac{2(n+1)(n-\omega)}{\omega(\omega+1)(\omega+2)}\alpha + \frac{3(n+1)(n-\omega)(n-\omega-1)}{\omega(\omega+1)(\omega+2)(\omega+3)} = 0$$

siue:

$$\beta - \frac{2(n-\omega+1)}{\omega+1}\alpha + \frac{3(n-\omega+1)(n-\omega)}{(\omega+1)(\omega+2)} = 0$$

$$\beta - \frac{2(n-\omega)}{\omega+2}\alpha + \frac{3(n-\omega)(n-\omega-1)}{(\omega+2)(\omega+3)} = 0$$

quarum aequationum differentia dat:

$$\frac{2(n+1)}{(\omega+1)(\omega+2)}\alpha - \frac{2\cdot 3(n+2)(n-\omega)}{(\omega+1)(\omega+2)(\omega+3)} = 0$$

hincque fit:

$$\alpha = \frac{3(n-\omega)}{\omega+3}; \quad \beta = \frac{3(n-\omega)(n-\omega+1)}{(\omega+3)(\omega+2)}; \text{ et } \gamma = \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+3)(\omega+2)(\omega+1)}$$

His autem valoribus pro denominatore inuentis pro numeratore reperientur:

$$A = \frac{\omega}{\omega+3} \cdot \frac{n+3}{1}$$

$$B = \frac{\omega(\omega-1)}{(\omega+3)(\omega+2)} \cdot \frac{(n+3)(n+2)}{1 \cdot 2}$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{(\omega+3)(\omega+2)(\omega+1)} \cdot \frac{(n+3)(n+2)(n+1)}{1 \cdot 2 \cdot 3}$$

$$D = \frac{(\omega-1)(\omega-2)(\omega-3)}{(\omega+3)(\omega+2)(\omega+1)} \cdot \frac{(n+3)(n+2)(n+1)n}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$E = \frac{(\omega-2)(\omega-3)(\omega-4)}{(\omega+3)(\omega+2)(\omega+1)} \cdot \frac{(n+3)(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$F = \frac{(\omega-3)(\omega-4)(\omega-5)}{(\omega+3)(\omega+2)(\omega+1)} \cdot \frac{(n+3)(n+2)(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

etc.

ac denominator formabitur ex his valoribus:

$$\alpha = \frac{3(n-\omega)}{\omega+3}$$

$$\beta = \frac{3(n-\omega)(n-\omega+1)}{(\omega+3)(\omega+2)}$$

$$\gamma = \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+3)(\omega+2)(\omega+1)}$$

quibus substitutis erit

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \text{etc.}}{1 - \alpha x + \beta x^2 - \gamma x^3}$$

## Coroll. I.

21. Manifestum hic est, quicunque numeri integer positiuus pro $\omega$ assumatur, in numerator semper terminos ternos successiuos in nihilum abire Ita si sit $\omega = 0$ erit

(1+x)

$$(1+x)^n = \frac{1 + * + * + * - \frac{(n+3)(n+2)(n+1)n}{1.\ 2.\ 3.\ 4} x^4 - \frac{4(n+3)(n+2)(n+1)n(n-1)}{1.\ 2.\ 3.\ 4.\ 5} x^5 - \text{etc.}}{1 - \frac{n}{1} x + \frac{n(n+1)}{1.\ 2} x^2 - \frac{n(n+1)(n+2)}{1.\ 2.\ 3} x^3}$$

Unde reiectis in numeratore terminis, qui post euanescentes sequuntur, erit proxime:

$$(1+x)^n = \frac{1}{1 - \frac{n}{1} x + \frac{n(n+1)}{1.\ 2} x^2 - \frac{n(n+1)(n+2)}{1.\ 2.\ 3} x^3}.$$

## Coroll. 2.

22. Simili modo ponendo $\omega = 1$ erit proxime:

$$(1+x)^n = \frac{1 + \frac{n+3}{4} x}{1 - \frac{3(n-1)}{4} x + \frac{3(n-1)n}{4.\ 2} x^2 - \frac{(n-1)n(n+1)}{4.\ 3.\ 2} x^3}$$

at si sumatur $\omega = 2$ erit

$$(1+x)^n = \frac{1 + \frac{2(n+3)}{5} x + \frac{(n+3)(n+2)}{5.\ 4} x^2}{1 - \frac{3(n-2)}{5} x + \frac{3(n-2)(n-1)}{5.\ 4} x^2 - \frac{(n-2)(n-1)n}{5.\ 4.\ 3} x^3}$$

posito vero $\omega = 3$ erit

$$(1+x)^n = \frac{1 + \frac{3(n+3)}{6} x + \frac{3(n+3)(n+2)}{6.\ 5} x^2 + \frac{(n+3)(n+2)(n+1)}{6.\ 5.\ 4} x^3}{1 - \frac{3(n-3)}{6} x + \frac{3(n-3)(n-2)}{6.\ 5} x^2 - \frac{(n-3)(n-2)(n-1)}{6.\ 5.\ 4} x^3}.$$

## Coroll. 3.

23. Postrema haec formula ideo est notatu digna, quod numerator et denominator pari terminorum numero constat, et quod alter in alterum abit, si exponens $n$ sumatur negatiue. Haec ergo expressio conferenda est cum similibus ex problematibus superioribus ortis:

$$(1+x)^n = \frac{1 + \frac{(n+1)}{2} x}{1 - \frac{(n-1)}{2} x} \ldots (\S.\ 7.)$$

$(1+x:$

$$(1+x)^n = \frac{1 + \frac{2(n+2)}{4}x + \frac{(n+2)(n+1)}{4 \cdot 3}x^2}{1 - \frac{2(n-2)}{4}x + \frac{(n-2)(n-1)}{4 \cdot 3}x^2} \ldots (\S.\ 17.)$$

vnde ſimul ordo huiusmodi formularum facile colligitur.

## Problema IV.

24. *Binomii poteſtatem $(1+x)^n$ transformare in huiusmodi progreſſionem maxime conuergentem:*

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}}{1 - \alpha x + \beta x^2 - \gamma x^3 + \delta x^4 - \varepsilon x^5 + \zeta x^6 - \text{etc.}}$$

*denominatore exiſtente multinomio quocunque.*

## Solutio.

Si ſolutiones praecedentium problematum conſulamus, leui attentione adhibita inde ſequentem ſolutionem generalem colligimus:

$$A = \frac{\omega}{\omega+\Phi} \cdot \frac{n+\Phi}{1}$$

$$B = \frac{\omega(\omega-1)}{(\omega+\Phi)(\omega+\Phi-1)} \cdot \frac{(n+\Phi)(n+\Phi-1)}{1 \cdot 2}$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{(\omega+\Phi)(\omega+\Phi-1)(\omega+\Phi-2)} \cdot \frac{(n+\Phi)(n+\Phi-1)(n+\Phi-2)}{1 \cdot 2 \cdot 3}$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{(\omega+\Phi)(\omega+\Phi-1)(\omega+\Phi-2)(\omega+\Phi-3)} \cdot \frac{(n+\Phi)(n+\Phi-1)(n+\Phi-2)(n+\Phi-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.

deinde vero pro denominatore:

$$\alpha = \frac{\Phi(n-\omega)}{1(\omega+\Phi)}$$

$$\beta = \frac{\Phi(\Phi-1)}{1 \cdot 2} \cdot \frac{(n-\omega)(n-\omega+1)}{(\omega+\Phi)(\Phi+\Phi-1)}$$

$$\gamma = \frac{\Phi(\Phi-1)(\Phi-2)}{1 \cdot 2 \cdot 3} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+\Phi)(\omega+\Phi-1)(\omega+\Phi-2)}$$

$$\delta = \frac{\Phi(\Phi-1)(\Phi-2)(\Phi-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)(n-\omega+3)}{(\omega+\Phi)(\omega+\Phi-1)(\omega+\Phi-2)(\omega+\Phi-3)}$$

etc.

qui

qui valores ad praecedentium formam propius reducuntur vt sit:

$$\alpha = \frac{\Phi}{\Phi + \omega} \cdot \frac{n - \omega}{1}$$

$$\beta = \frac{\Phi(\Phi - 1)}{(\Phi + \omega)(\Phi + \omega - 1)} \cdot \frac{(n - \omega)(n - \omega + 1)}{1 \cdot 2}$$

$$\gamma = \frac{\Phi(\Phi - 1)(\Phi - 2)}{(\Phi + \omega)(\Phi + \omega - 1)(\Phi + \omega - 2)} \cdot \frac{(n - \omega)(n - \omega + 1)(n - \omega + 2)}{1 \cdot 2 \cdot 3}$$

$$\delta = \frac{\Phi(\Phi - 1)(\Phi - 2)(\Phi - 3)}{(\Phi + \omega)(\Phi + \omega - 1)(\Phi + \omega - 2)(\Phi + \omega - 3)} \cdot \frac{(n - \omega)(n - \omega + 1)(n - \omega + 2)(n - \omega + 3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.

Etsi autem ex hac lege etiam denominator in infinitum continuari possit; tamen ex principio, vnde eum deduximus, patet eum non vltra terminos euanescentes produci debere, siquidem pro $\Phi$ sumatur numerus positiuus integer.

## Coroll. 1.

25. Denominator ergo ex numeratore formari potest, si numeri $\Phi$ et $\omega$ inter se permutentur, et loco $n$ scribatur $-n$. At posito $-n$ pro $+n$ formula $(1+x)^n$ abit in $(1+x)^{-n}$; vnde si fuerit $(1+x)^n = \frac{P}{Q}$ erit $(1+x)^{-n} = \frac{Q}{P}$, ex quo ratio huius conuersionis eo clarius perspicitur.

## Coroll. 2.

26. Cum igitur numerator et denominator inter se permutari possint, etiam numeratorem apud terminos euanescentes abrumpere licet; tum vero denominatorem in infinitum continuari oportet, vt fractio obtineatur potestati $(1+x)^n$ aequalis.

 Coroll.

## Coroll. 3.

27. Si ſumatur $\Phi = \omega$, numerator et denominator multo magis inter ſe aſſimilantur, ac tantum ratione ſigni exponentis $n$ a ſe inuicem diſcrepabunt. Erit autem tunc:

$$A = \frac{\omega}{2\omega} \cdot \frac{n+\omega}{1}$$

$$B = \frac{\omega(\omega-1)}{2\omega(2\omega-1)} \cdot \frac{(n+\omega)(n+\omega-1)}{1 \cdot 2}$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{2\omega(2\omega-1)(2\omega-2)} \cdot \frac{(n+\omega)(n+\omega-1)(n+\omega-2)}{1 \cdot 2 \cdot 3}$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{2\omega(2\omega-1)(2\omega-2)(2\omega-3)} \cdot \frac{(n+\omega)(n+\omega-1)(n+\omega-2)(n+\omega-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.

$$\alpha = \frac{\omega}{2\omega} \cdot \frac{n-\omega}{1}$$

$$\beta = \frac{\omega(\omega-1)}{2\omega(2\omega-1)} \cdot \frac{(n-\omega)(n-\omega+1)}{1 \cdot 2}$$

$$\gamma = \frac{\omega(\omega-1)(\omega-2)}{2\omega(2\omega-1)(2\omega-2)} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{1 \cdot 2 \cdot 3}$$

$$\delta = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{2\omega(2\omega-1)(2\omega-2)(2\omega-3)} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)(n-\omega+3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.

## Coroll. 4.

28. Hinc formulae ſuperiores (23) ad approximandum perquam idoneae deriuantur:

$$(1+x)^n = \frac{1 + \frac{1}{2} \cdot \frac{n+1}{1} x}{1 - \frac{1}{2} \cdot \frac{n-1}{1} x}$$

$$(1+x)^n = \frac{1 + \frac{2}{4} \cdot \frac{n+2}{1} x + \frac{2 \cdot 1}{4 \cdot 3} \cdot \frac{(n+2)(n+1)}{1 \cdot 2} x^2}{1 - \frac{2}{4} \cdot \frac{n-2}{1} x + \frac{2 \cdot 1}{4 \cdot 3} \cdot \frac{(n-2)(n-1)}{1 \cdot 2} x^2}$$

$$(1+x)^n = \frac{1 + \frac{3}{6}\cdot\frac{(n+3)}{1}x + \frac{3\cdot 2}{6\cdot 5}\cdot\frac{(n+3)(n+2)}{1\cdot\ \ 2}x^2 + \frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{(n+3)(n+2)(n+1)}{1\cdot\ \ 2\cdot\ \ 3}x^3}{1 - \frac{3}{6}\cdot\frac{(n-3)}{1}x + \frac{3\cdot 2}{6\cdot 5}\cdot\frac{(n-3)(n-2)}{1\cdot\ \ 2}x^2 - \frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{(n-3)(n-2)(n-1)}{1\cdot\ \ 2\cdot\ \ 3}x^3}$$

quae quomodo vlterius continuari debeant sponte patet.

## Scholion 1.

29. Hae formulae eo magis sunt notatu dignae, quo minus earum ratio patet; nam etsi tam in numeratore quam denominatore lex progressionis est perspicua, secundum quam vterque in infinitum continuatur, tamen iam animaduertimus, alterutrum tantum in infinitum produci oportere, altero ex finito terminorum numero constante, ibi scilicet quouis casu terminari debet, vbi termini aliquot euanescere incipiunt; etiamsi deinceps iterum termini finitae magnitudinis occurrant. Haec autem ita sunt interpretanda, si in valoribus litterarum A, B, C, etc. $\alpha$, $\beta$, $\gamma$ etc. factor numeratoris euanescens a factore denominatoris euanescente tolli censeatur, ita vt fractio $\frac{\omega - m}{2\omega - 2m}$ casu $\omega = m$ vnitati aequalis statuatur. Sin autem, vti calculi ratio exigit, haec fractio tantum semissi vnitatis aequalis capiatur, tum continui ratio non amplius infringitur; ac si hac lege recepta tam numerator quam denominator etiam vltra terminos euanescentes in infinitum continuatur, fractio resultans formulae $(1+x)^n$ perfecte erit aequalis. Quod idem in genere est tenendum, dummodo inter numeros $\Phi$ et $\omega$ certa ratio statuatur, ita vt si $\Phi = \lambda\,\omega$ fractionis $\frac{\omega - m}{(\lambda+1)\omega - (\lambda+1)m}$

 etiam

etiam casu $\omega = m$ sumatur $= \frac{1}{\lambda + 1}$ ex quo haec cautio neutiquam principio continuitatis aduersari est putanda.

## Scholion 2.

30. Quo haec clarius perspiciantur, consideremus casum $\omega = 0$, et ob $\frac{\omega}{2\omega} = \frac{1}{2}$, erit numerator nostrae fractionis:

$$1 + \tfrac{1}{2}\cdot\tfrac{n}{1}x + \tfrac{1}{2}\cdot\tfrac{n(n-1)}{1\cdot 2}x^2 + \tfrac{1}{2}\cdot\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \tfrac{1}{2}\cdot\tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \text{etc.}$$

et denominator:

$$1 - \tfrac{1}{2}\cdot\tfrac{n}{1}x + \tfrac{1}{2}\cdot\tfrac{n(n+1)}{1\cdot 2}x^2 - \tfrac{1}{2}\cdot\tfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \tfrac{1}{2}\cdot\tfrac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}x^4 - \text{etc.}$$

manifestum autem est numeratoris valorem esse $= \frac{1}{2} + \frac{1}{2}(1 + x)^n$ denominatoris vero $= \frac{1}{2} + \frac{1}{2}(1 + x)^{-n}$, illumque ergo per hunc diuisum praebere $(1 + x)^n$. Simili modo si ponatur $\omega = 1$, erit

| pro numeratore | pro denominatore |
|---|---|
| $A = \frac{1}{2}\cdot\frac{n+1}{1}$ | $\alpha = \frac{1}{2}\cdot\frac{n-1}{1}$ |
| $B = 0$ | $\beta = 0$ |
| $C = -\frac{1}{4}\cdot\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}$ | $\gamma = -\frac{1}{4}\cdot\frac{(n-1)n(n+1)}{1\cdot 2\cdot 3}$ |
| $D = -\frac{2}{4}\cdot\frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}$ | $\delta = -\frac{2}{4}\cdot\frac{(n-1)n(n+1)(n+2)}{1\cdot 2\cdot 3\cdot 4}$ |
| $E = -\frac{3}{4}\cdot\frac{(n+1)n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$ | $\varepsilon = -\frac{3}{4}\cdot\frac{(n-1)n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$ |
| etc. | etc. |

atque hinc colligitur fore

$$\text{numeratorem} = \tfrac{1}{2} + \tfrac{1}{4}(n+1)x + \left(\tfrac{1}{2} - \tfrac{1}{4}(n-1)x\right)(1+x)^n$$
$$\text{denominatorem} = \tfrac{1}{2} - \tfrac{1}{4}(n-1)x + \left(\tfrac{1}{2} + \tfrac{1}{4}(n+1)x\right)(1+x)^{-n}$$

quorum ille per hunc diuisus manifesto praebet formulam propositam $(1 + x)^n$. Sin autem in denominatore termini litteris $\gamma, \delta, \varepsilon$ etc. affecti omitterenterentur, tum in numeratore loco fractionis $\frac{\omega-1}{2\omega-2}=\frac{1}{2}$ vnitas statui deberet ob legem supra stabilitam, vnde valores C, D, E etc. duplo prodirent maiores; foretque numeratoris valor $=(1-\frac{1}{2}(n-1)x)(1+x)^n$, denominator vero $=1-\frac{1}{2}(n-1)x$ qua fractione iterum veritas obtinetur. Videamus ergo, quomodo per huiusmodi formulas tam quantitates radicales, quam exponentiales et logarithmi commode vero proxime exhiberi queant; quando quidem constat tam logarithmos quam exponentiales quantitates ad formam $(1+x)^n$ reuocari posse.

## Problema V.

31. Radicem quadratam ex quouis numero nonquadrato proposito per formulas ante exhibitas proxime assignare.

## Solutio.

Sit numerus propositus non quadratus $= aa+b$, et ponatur $\frac{b}{aa}=x$ erit $aa+b=aa(1+x)$ ideoque $\sqrt{(aa+b)}=a(1+x)^{\frac{1}{2}}$. Habebimus ergo $n=\frac{1}{2}$, et ex praecedente problemate formulae continuo magis ad $\sqrt{(aa+b)}$ appropinquantes erunt:

$$\sqrt{(aa+b)}=\frac{1+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{b}{a^2}}{1+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{b}{a^2}}\,a$$

$$\sqrt{(aa+b)}=\frac{1+\frac{2}{4}\cdot\frac{5}{2}\cdot\frac{b}{a^2}+\frac{2\cdot1}{4\cdot3}\cdot\frac{5\cdot3}{2\cdot4}\cdot\frac{b^2}{a^4}}{1+\frac{2}{4}\cdot\frac{3}{2}\cdot\frac{b}{a^2}+\frac{2\cdot1}{4\cdot3}\cdot\frac{3\cdot1}{2\cdot4}\cdot\frac{b^2}{a^4}}\,a$$

V 3 $\sqrt{(aa}$

$$\sqrt{(aa+b)} = \frac{1 + \frac{3}{6} \cdot \frac{7}{2} \cdot \frac{b}{a^2} + \frac{3 \cdot 2}{6 \cdot 5} \cdot \frac{7 \cdot 5}{2 \cdot 4} \cdot \frac{b^2}{a^4} + \frac{3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4} \cdot \frac{7 \cdot 5 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{b^3}{a^6}}{1 + \frac{3}{6} \cdot \frac{5}{2} \cdot \frac{b}{a^2} + \frac{3 \cdot 2}{6 \cdot 5} \cdot \frac{5 \cdot 3}{2 \cdot 4} \cdot \frac{b^2}{a^4} + \frac{3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4} \cdot \frac{5 \cdot 3 \cdot 1}{2 \cdot 4 \cdot 6} \cdot \frac{b^3}{a^6}} a$$

etc.

Euolutis autem his factoribus et posito breuitatis ergo $\frac{b}{aa} = x$ consequemur formas sequentes:

$$\sqrt{(aa+b)} = \frac{1 + \frac{3}{4}x}{1 + \frac{1}{4}x} a$$

$$\sqrt{(aa+b)} = \frac{1 + \frac{5}{4}x + \frac{5}{16}xx}{1 + \frac{3}{4}x + \frac{1}{16}xx} a$$

$$\sqrt{(aa+b)} = \frac{1 + \frac{7}{4}x + \frac{7}{8}xx + \frac{7}{64}x^3}{1 + \frac{5}{4}x + \frac{3}{8}xx + \frac{1}{64}x^3} a$$

$$\sqrt{(aa+b)} = \frac{1 + \frac{9}{4}x + \frac{27}{16}xx + \frac{15}{32}x^3 + \frac{9}{256}x^4}{1 + \frac{7}{4}x + \frac{15}{16}xx + \frac{5}{32}x^3 + \frac{1}{256}x^4} a$$

$$\sqrt{(aa+b)} = \frac{1 + \frac{11}{4}x + \frac{11}{4}xx + \frac{77}{64}x^3 + \frac{55}{256}x^4 + \frac{11}{1024}x^5}{1 + \frac{9}{4}x + \frac{7}{4}xx + \frac{35}{64}x^3 + \frac{15}{256}x^4 + \frac{1}{1024}x^5} a$$

etc.

Sin autem ponamus $\frac{x}{4} = y$ seu $y = \frac{b}{4aa}$ erit

$$\sqrt{(aa+b)} = \frac{1 + 3y}{1 + y} a$$

$$\sqrt{(aa+b)} = \frac{1 + 5y + 5yy}{1 + 3y + yy} a$$

$$\sqrt{(aa+b)} = \frac{1 + 7y + 14yy + 7y^3}{1 + 5y + 6yy + y^3} a$$

$$\sqrt{(aa+b)} = \frac{1 + 9y + 27yy + 30y^3 + 9y^4}{1 + 7y + 15yy + 10y^3 + y^4} a$$

$$\sqrt{(aa+b)} = \frac{1 + 11y + 44yy + 77y^3 + 55y^4 + 11y^5}{1 + 9y + 28yy + 35y^3 + 15y^4 + y^5} a$$

etc.

Coroll.

## Coroll. 1.

32. Si formularum harum numeratores et denominatores attentius contemplemur, non difficulter obseruabimus, vtrosque constituere progressionem recurrentem secundi ordinis, et quemlibet terminum ita dependere a binis praecedentibus, vt si terni termini ordine sint P, Q, R, semper sit $R = (1 + 2y)Q - yyP$ seu scala relationis habeatur $1 + 2y, - yy$.

## Coroll. 2.

33. Si pro $y$ statuamus valorem $\frac{b}{4aa}$, et numeratorem denominatoremque a fractionibus liberemus, habebimus sequentes formulas:

$$\sqrt{(aa+b)} = \frac{4a^2 + 3b}{4a^2 + b}\, a$$

$$\sqrt{(aa+b)} = \frac{16a^4 + 20a^2b + 5bb}{16a^4 + 12a^2b + bb}\, a$$

$$\sqrt{(aa+b)} = \frac{64a^6 + 112a^4b + 56a^2b^2 + 7b^3}{64a^6 + 80a^4b + 24a^2b^2 + b^3}\, a$$

$$\sqrt{(aa+b)} = \frac{256a^8 + 576a^6b + 432a^4b^2 + 120a^2b^3 + 9b^4}{256a^8 + 448a^6b + 240a^4b^2 + 40a^2b^3 + b^4}$$

etc.

## Coroll. 3.

34. In his formulis iterum tam numeratores quam denominatores seriem constituunt recurrentem, cuius scala relationis est $2(2aa + b), - bb$, ita vt, si P, Q, R denotent tres terminos se inuicem excipientes, futurum sit

$$R = 2(2aa + b)Q - bbP.$$

At

At seriei numeratorum duo termini initiales sunt 1, et $4aa+3b$ denominatorum vero 1 et $4aa+b$, vnde reliqui facile reperiuntur.

## Coroll. 4.

35. Si fractio $\frac{b}{4aa}$ ad minores terminos reduci potest, his potius loco ipsorum $b$ et $4aa$ vti conueniet. Ponatur ergo in minimis terminis: $\frac{b}{4aa}=\frac{y}{z}$, atque habebimus:

$$\sqrt{(aa+b)}=\frac{z+3y}{z+y}\,a$$

$$\sqrt{(aa+b)}=\frac{zz+5yz+5yy}{zz+3yz+yy}\,a$$

$$\sqrt{(aa+b)}=\frac{z^3+7yz^2+14y^2z+7y^3}{z^3+5yz^2+6y^2z+y^3}\,a$$

hincque erit $R=(z+2y)Q-yyP$.

## Coroll. 5.

36. Hae fractiones adhuc commodius exprimi possunt hoc modo:

$$\sqrt{(aa+b)}=\frac{z+2y+y\cdot 1}{z+2y-y\cdot 1}\,a$$

$$\sqrt{(aa+b)}=\frac{z^2+4yz+3yy+y(z+2y)}{z^2+4yz+3yy-y(z+2y)}\,a$$

$$\sqrt{(aa+b)}=\frac{z^3+6yz^2+10y^2z+4y^3+y(z^2+4yz+3y^2)}{z^3+6yz^2+10y^2z+4y^3-y(z^2+4yz+3y^2)}\,a$$

$$\sqrt{(aa+b)}=\frac{z^4+8yz^3+21y^2z^2+20y^3z+5y^4+y(z^3+6yz^2+10y^2z+4y^3)}{z^4+8yz^3+21y^2z^2+20y^3z+5y^4-y(z^3+6yz^2+10y^2z+4y^3)}\,a$$

etc.

## Coroll. 6.

37. Pro his fractionibus formandis sufficit vnicam hanc seriem constituisse:

$1,\ z+2y;\ z^2+4yz+3y^2;\ z^3+6yz^2+10y^2z+4y^3;\ \ldots P;\ Q;\ R$

quae

quae pariter est recurrens ad legem $R = (z + 2y)Q - yyP$. Formata autem hac serie erit proxime $\sqrt{(aa+b)} = \frac{Q+yP}{Q-yP}\,a$: quae scilicet fractio ex binis terminis se immediate sequentibus illius seriei facillime formatur.

## Exemplum 1.

38. *Radicem quadratam ex 2 proxime exhibere.*

Cum sit $aa+b=2$ erit $a=1$ et $b=1$, vnde $\frac{b}{4aa} = \frac{1}{4} = \frac{y}{z}$; ergo $y=1$ et $z=4$, atque $z+2y=6$. Quare ex scala relationis $R = 6Q - P$ formetur haec series recurrens:

$1;\ 6;\ 35;\ 204;\ 1189;\ 6930;\ 40391 \ldots P, Q, R$

et fractiones $\frac{Q+P}{Q-P}$ ad $\sqrt{2}$ continuo magis appropinquantes sunt:

$\sqrt{2} = \frac{7}{5};\ \frac{41}{29};\ \frac{239}{169};\ \frac{1393}{985};\ \frac{8119}{5741};\ \frac{47321}{33461}.$

## Exemplum 2.

39. *Radicem quadratam ex 3 proxime exhibere.*

Cum sit $aa+b=3$, statuatur $a=1$, erit $b=2$; et $\frac{b}{4aa} = \frac{2}{4} = \frac{1}{2}$ vnde fit $y=1$ et $z=2$; ergo $z+2y=4$. Quare ex scala relationis $R=4Q-P$ formetur haec series recurrens:

$1;\ 4;\ 15;\ 56;\ 209;\ 780;\ 2911;\ 10864 \ldots P, Q, R$

eritque proxime $\sqrt{3} = \frac{Q+P}{Q-P}$, siue

$\sqrt{3} = \frac{5}{3};\ \frac{19}{11};\ \frac{71}{41};\ \frac{265}{153};\ \frac{989}{571};\ \frac{3691}{2131};\ \frac{13775}{7953}$ etc.

*Aliter.* Vel statuamus $a=2$; vt sit $b=-1$; erit $\frac{b}{4aa} = -\frac{1}{16} = \frac{y}{z}$ vnde $y=-1$; $z=16$ et $z+2y=14$.

 Quare

Quare ex ſcala relationis; $R = 14Q - P$ formetur ſeries recurrens:

$$1;\ 14;\ 195;\ 2716;\ 37829;\ 526890 \ldots P,\ Q,\ R$$

eritque proxime $\sqrt{3} = \frac{Q-P}{Q+P}\, 2$ ſiue

$$\sqrt{3} = \tfrac{13}{15}.\,2;\ \tfrac{181}{209}.\,2;\ \tfrac{2521}{2911}.\,2;\ \tfrac{35113}{40545}.\,2;\ \text{etc. vel}$$

$$\sqrt{3} = \tfrac{26}{15};\ \tfrac{362}{209};\ \tfrac{5042}{2911};\ \tfrac{70226}{40545};\ \text{etc.}$$

## Problema VI.

40. *Radicem cubicam ex quouis numero non-cubo propoſito per formulas ante exhibitas proxime aſſignare.*

## Solutio.

Sit numerus propoſitus non cubus $= a^3 + b$, et ponatur $\frac{b}{a^3} = x$ erit $a^3 + b = a^3(1+x)$ ideoque $\sqrt[3]{(a^3+b)} = a(1+x)^{\frac{1}{3}}$. Habemus ergo $n = \frac{1}{3}$; vnde ex §. 28. nanciſcemur has approximationes:

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1 + \frac{1}{2}.\frac{4}{3}x}{1 + \frac{1}{2}.\frac{2}{3}x}$$

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1 + \frac{2}{4}.\frac{7}{3}x + \frac{2.1}{4.3}.\frac{7.4}{3.3}xx}{1 + \frac{2}{4}.\frac{5}{3}x + \frac{2.1}{4.3}.\frac{5.2}{3.3}xx}$$

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1 + \frac{3}{6}.\frac{10}{3}x + \frac{3.2}{6.5}.\frac{10.7}{3.3}xx + \frac{3.2.1}{6.5.4}.\frac{10.7.4}{3.3.3}x^3}{1 + \frac{3}{6}.\frac{8}{3}x + \frac{3.2}{6.5}.\frac{8.5}{3.3}xx + \frac{3.2.1}{6.5.4}.\frac{8.5.2}{3.3.3}x^3}$$

etc.

Enoſta-

Euolutis autem his coefficientibus habebimus:

$$\sqrt[3]{(a^3+b)} = a.\frac{1+\frac{2}{3}x}{1+\frac{1}{3}x}$$

$$\sqrt[3]{(a^3+b)} = a.\frac{1+\frac{7}{6}x+\frac{7}{27}xx}{1+\frac{5}{6}x+\frac{5}{54}xx}$$

$$\sqrt[3]{(a^3+b)} = a.\frac{1+\frac{5}{3}x+\frac{7}{9}xx+\frac{7}{81}x^3}{1+\frac{4}{3}x+\frac{4}{9}xx+\frac{2}{81}x^3}$$

etc.

## Coroll. 1.

41. Si loco $x$ valorem $\frac{b}{a^3}$ substituamus, et fractiones implicatas tollamus, obtinebimus formulas sequentes:

$$\sqrt[3]{(a^3+b)} = \frac{3a^3+2b}{3a^3+b}a$$

$$\sqrt[3]{(a^3+b)} = \frac{54a^6+63a^3b+14bb}{54a^6+45a^3b+5bb}a$$

$$\sqrt[3]{(a^3+b)} = \frac{81a^9+135a^6b+63a^3b^2+7b^3}{81a^9+108a^6b+36a^3b^2+2b^3}a$$

etc.

vbi autem commodam progressionis legem definire non licet.

## Coroll. 2.

42. Sufficit autem, forma vti priori; inde enim cubus ad numerum propositum propius accedens colligitur, cuius radix pro $a$ posita nouum

X 2 dabit

dabit valorem pro $b$ et $x$. Sic si radix cubica ex 2 quaeratur, erit statim $a = 1$, et proxime $\sqrt[3]{2} = \frac{5}{4}$. Sit iam $a = \frac{5}{4}$; et fit $b = 2 - a^3 = \frac{3}{64}$; et $x = \frac{3}{125}$; vnde erit denuo per formam priorem:

$$\sqrt[3]{2} = \frac{125 + 2}{125 + 1} \cdot \frac{5}{4} = \frac{127}{126} \cdot \frac{5}{4} = \frac{635}{504}$$

cuius fractionis cubus est $2 - \frac{253}{128024064}$, qui ergo a veritate tantum parte $\frac{1}{506024}$ deficit.

## Coroll. 3.

43. Simili modo formulae pro extractione radicum altiorum potestatum formari possunt. Ita si quaeratur $\sqrt[m]{(a^m + b)}$, ponatur $x = \frac{b}{a^m}$ et $n = \frac{1}{m}$, hincque habebitur:

$$\sqrt[m]{(a^m + b)} = \frac{2ma^m + (m+1)b}{2ma^m + (m-1)b}\, a$$

quae etiam sufficere potest ad radices quantumuis exacte definiendas.

## Problema VII.

44. *Per formulas supra inuentas proxime exprimere logarithmum cuiusque numeri propositi.*

## Solutio.

Sit $1 + x$ numerus propositus, et constat eius logarithmum hyperbolicum esse $l(1+x) = \frac{(1+x)^n - 1}{n}$

existen-

existente $n = \infty$. Quodsi iam in formulis supra inventis $n$ spectemus vt numerum infinite paruum; habebimus:

$$(1+x)^n = \frac{1+\frac{1}{2}(1+n)x}{1+\frac{1}{2}(1-n)x}$$

$$(1+x)^n = \frac{1+\frac{2}{4}(2+n)x+\frac{2\cdot 1}{4\cdot 3}(1+\frac{3}{2}n)xx}{1+\frac{2}{4}(2-n)x+\frac{2\cdot 1}{4\cdot 3}(1-\frac{3}{2}n)xx}$$

$$(1+x)^n = \frac{1+\frac{3}{6}(3+n)x+\frac{3\cdot 2}{6\cdot 5}(3+\frac{5}{2}n)x^2+\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}(1+\frac{11}{6}n)x^3}{1+\frac{3}{6}(3-n)x+\frac{3\cdot 2}{6\cdot 5}(3-\frac{5}{2}n)x^2+\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}(1-\frac{11}{6}n)x^3}$$

$$(1+x)^n = \frac{1+\frac{4}{8}(4+n)x+\frac{4\cdot 3}{8\cdot 7}(6+\frac{7}{2}n)x^2+\frac{4\cdot 3\cdot 2}{8\cdot 7\cdot 6}(4+\frac{13}{3}n)x^3+\frac{4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5}(1+\frac{25}{12}n)x^4}{1+\frac{4}{8}(4-n)x+\frac{4\cdot 3}{8\cdot 7}(6-\frac{7}{2}n)x^2+\frac{4\cdot 3\cdot 2}{8\cdot 7\cdot 6}(4-\frac{13}{3}n)x^3+\frac{4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5}(1-\frac{25}{12}n)x^4}$$

etc.

Quodsi iam hic ponatur $n = 0$, habebimus pro $l(1+x)$ sequentes approximationes:

$$l(1+x) = \frac{x}{1+\frac{1}{2}x}$$

$$l(1+x) = \frac{x+\frac{1}{2}xx}{1+x+\frac{1}{6}xx}$$

$$l(1+x) = \frac{x+xx+\frac{11}{60}x^3}{1+\frac{3}{2}x+\frac{3}{5}x^2+\frac{1}{20}x^3}$$

$$l(1+x) = \frac{x+\frac{3}{2}xx+\frac{13}{21}x^3+\frac{5}{84}x^4}{1+2x+\frac{9}{7}xx+\frac{2}{7}x^3+\frac{1}{70}x^4}$$

etc.

Vel si ponatur $x = \frac{m}{n}$, quoniam fractionum loga-

 rithmos

rithmos potissimum indagare conuenit, et fractiones partiales tollantur, fiet

$$l(1+\tfrac{m}{n}) = \frac{2m}{2n+m}$$

$$l(1+\tfrac{m}{n}) = \frac{6mn+3mm}{6nn+6mn+mm}$$

$$l(1+\tfrac{m}{n}) = \frac{60mn^2+60m^2n+11m^3}{60n^3+90mn^2+36m^2n+3m^3}$$

$$l(1+\tfrac{m}{n}) = \frac{420mn^3+630m^2n^2+260m^3n+25m^4}{420n^4+840mn^3+540m^2n^2+120m^3n+6m^4}$$

etc.

haeque fractiones tam prope accedunt ad verum valorem $l(1+\frac{m}{n})$, vt seriei vulgaris

$$l(1+\tfrac{m}{n}) = \frac{m}{n} - \frac{m^2}{2n^2} + \frac{m^3}{3n^3} - \frac{m^4}{4n^4} \text{ etc.}$$

ingens terminorum numerus capi deberet ad parem approximationem obtinendam.

## Coroll. 1.

45. Ita si logarithmum hyperbolicum binarii desideremus, ob $m=1$ et $n=1$, sequentes prodibunt approximationes:

$$l\,2 = \tfrac{2}{3};\ \tfrac{9}{13};\ \tfrac{131}{189};\ \tfrac{3526}{5087};\ [\ldots]$$

quibus fractionibus in decimales conuersis, cum sit

$$l\,2 = 0,693147180559945\,3$$

erit proxime

$l\,2 = 0,666666$

$l\,2 = 0,692307$

$l\,2 = 0,693121$

$l\,2 = 0,69314635$

vere $l\,2 = 0,69314718$

sicque

sicque quarta fractio a veritate tantum parte $\frac{83}{100000000}$ deficit.

## Coroll. 2.

46. Numerorum autem binario minorum logarithmi multo adhuc exactius reperiuntur. Ita cum sit $l\frac{3}{2} = 0,405465108108164$ ponamus $m = 1$ et $n = 2$, nostraeque formulae dabunt proxime

$$l\frac{3}{2} = \frac{2}{5} = 0,40000000$$
$$l\frac{3}{2} = \frac{15}{37} = 0,40540540 5$$
$$l\frac{3}{2} = \frac{371}{915} = 0,405464481$$
$$l\frac{3}{2} = \frac{6425}{15846} = 0,4054651016$$

error scilicet huius vltimae fractionis est $\frac{65}{10000000000}$ ideoque plus quam centies minor quam casu praecedente.

## Coroll. 3.

47. Quando ergo fractio $\frac{m}{n}$ adeo semisse est minor, tum erit tam exacte

$$l\left(1 + \frac{m}{n}\right) = \frac{420\,m n^3 + 630\,m^2 n^2 + 260\,m^3 n + 25\,m^4}{420\,n^4 + 840\,m n^3 + 540\,m^2 n - 120\,m^3 n + 6\,m^4}$$

vt error in fractione decimali post decimam demum notam percipiatur. Aliis autem methodis vix tam facile ad veritatem appropinquare licet.

## Coroll. 4.

48. Si fractio $\frac{m}{n}$ fuerit valde parua, tum sufficiet vel prima vel secunda formula, ita si $\frac{m}{n} = \frac{1}{8}$; prima formula dat $l\frac{9}{8} = \frac{2}{17} = 0,11764$, et secunda;

$$l\frac{9}{8} = \frac{51}{433} = 0,11778291$$ at reuera est
$$l\frac{9}{8} = 0,11778303$$

vnde

vnde fecunda formula circiter $\frac{x}{10000000}$ a veritate deficit.

## Problema VIII.

49. *Quantitatem exponentialem* $e^x$ *per formulas inuentas proxime exprimere, exiftente e numero, cuius logarithmus hyperbolicus aequatur vnitati.*

## Solutio.

Notum eft effe $e^x = (1 + \frac{x}{n})^n$, fi pro $n$ fumatur numerus infinitus. Scribamus ergo in formulis §. 28, $\frac{x}{n}$ loco $x$ et fimul ponamus $n = \infty$; atque obtinebimus fequentes approximationes

$$e^x = \frac{1 + \frac{1}{2}x}{1 - \frac{1}{2}x}$$

$$e^x = \frac{1 + \frac{2}{4}x + \frac{1}{4\cdot 3}xx}{1 - \frac{2}{4}x + \frac{1}{4\cdot 3}xx}$$

$$e^x = \frac{1 + \frac{3}{6}x + \frac{3}{6\cdot 5}xx + \frac{1}{6\cdot 5\cdot 4}x^3}{1 - \frac{3}{6}x + \frac{3}{6\cdot 5}xx - \frac{1}{6\cdot 5\cdot 4}x^3}$$

$$e^x = \frac{1 + \frac{4}{8}x + \frac{6}{8\cdot 7}xx + \frac{4}{8\cdot 7\cdot 6}x^3 + \frac{1}{8\cdot 7\cdot 6\cdot 5}x^4}{1 - \frac{4}{8}x + \frac{6}{8\cdot 7}xx + \frac{4}{8\cdot 7\cdot 6}x^3 + \frac{1}{8\cdot 7\cdot 6\cdot 5}x^4}$$

etc.

vnde lex, qua fequentes hujusmodi formulae confici debent, eft manifefta. Si fractiones partiales tollere velimus, habebimus:

$e^x =$

$$e^x = \frac{2+x}{2-x}$$

$$e^x = \frac{12+6x+xx}{12-6x+xx}$$

$$e^x = \frac{120+60x+12xx+x^3}{120-60x+12xx-x^3}$$

$$e^x = \frac{1680+840x+180xx+20x^3+x^4}{1680-840x+180xx-20x^3+x^4}$$

## Coroll. 1.

50. Hinc ergo erit ipſe numerus $e$ in fractionibus proximis:

$e = \frac{3}{1};\ \frac{19}{7};\ \frac{193}{71};\ \frac{2721}{1001};$ etc.

quarum fractionum hanc legem obſeruari conuenit, vt ſi ponatur:

$e = \frac{A}{\mathfrak{A}};\ \frac{B}{\mathfrak{B}};\ \frac{C}{\mathfrak{C}};\ \frac{D}{\mathfrak{D}};\ \frac{E}{\mathfrak{E}}$ etc. ſit

$A=3;\ B=6A+1;\ C=10B+A;\ D=14C+B;\ E=18D+C;$ etc.
$\mathfrak{A}=1;\ \mathfrak{B}=6\mathfrak{A}+1;\ \mathfrak{C}=10\mathfrak{B}+\mathfrak{A};\ \mathfrak{D}=14\mathfrak{C}+\mathfrak{B};\ \mathfrak{E}=18\mathfrak{D}+\mathfrak{C};$ etc.
vbi multiplicatores 6, 10, 14, 18, etc. ſunt numeri impariter pares.

## Coroll. 2.

51. Cum igitur ſit $e = 2,718281828459045235 36$ videamus quam prope fractiones inuentae accedant ad veritatem:

$e = \frac{3}{1} = 3,0000$
$e = \frac{19}{7} = 2,714285714$
$e = \frac{193}{71} = 2,718309859$
$e = \frac{2721}{1001} = 2,718281718$
etc.

vbi primi in partibus decimis, fecunda in millefimis, tertia in centies millefimis, et quarta in centies centenis millefimis aberrat.

## Coroll. 3.

52. Talis lex progreffionis etiam in formulis generalibus pro $e^x$ deprehenditur. Si enim noftras fractiones ponamus:

$$e^x = \frac{A}{\mathfrak{A}};\ \frac{B}{\mathfrak{B}};\ \frac{C}{\mathfrak{C}};\ \frac{D}{\mathfrak{D}};\ \frac{E}{\mathfrak{E}} \text{ etc. fumtis } A=1 \text{ et } \mathfrak{A}=1, \text{ erit:}$$

$$B=2+x;\ C=6B+Axx;\ D=10C+Bxx;\ E=14D+Cxx; \text{ etc.}$$
$$\mathfrak{B}=2-x;\ \mathfrak{C}=6\mathfrak{B}+\mathfrak{A}xx;\ \mathfrak{D}=10\mathfrak{C}+\mathfrak{B}xx;\ \mathfrak{E}=14\mathfrak{D}+\mathfrak{C}xx; \text{ etc.}$$

vnde feries tam numeratorum, quam denominatorum facile continuatur.

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SOLV-